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Numerical Solution of Extended Korteweg-de Vries Equation with Cubic Nonlinear Term and Fifth-Order Dispersion Term

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Abstract

This article addresses the problem that the Korteweg-de Vries (KdV) equation does not fully capture the complexity of nonlinear waves. To address this issue, we solve the extended Korteweg-de Vries (eKdV) equation, which includes the higher-order nonlinear and dispersion terms. The main objective is to investigate how cubic nonlinearity and fifth-order dispersion terms affect solitary waves propagation. A unique aspect of this study is the use of the Pseudospectral (PS) method, which allows for much longer numerical simulations compared to the previous studies without any existing the higher simulations frequencies. The results show that the Gardner equation, which dominant with nonlinear waves, leads to steepening and breaking of solitary waves. In contrast, the Kawahara equation, which reflects dispersive waves, exhibits instability and produces oscillatory tails. These findings provide valuable insights into the behavior of solitary waves and highlight the effectiveness of the Pseudospectral method in studying complex wave phenomena.

Keywords: pseudospectral method; extended Korteweg-de Vries equation; cubic nonlinear term; fifth-order dispersion term; Gardner equation; Kawahara equation.

1 Introduction

Nonlinear partial differential equations are important in many fields of science and engineering, such as chemical physics, optical fibre, plasma physics, fluid dynamics and many more [27, 29]. In fact, the eKdV equation belongs to the class of PDEs and has gained much more popularity as it models several physical phenomena in nature [31]. Khusnutdinova et al. [18] stated that the eKdV equation, also known as Gardner-Kawahara equation, can be written as,

$$u_t + \alpha u u_x + \alpha_1 u^2 u_x + \beta u_{xxx} + \beta_1 u_{xxxxx} = 0, \tag{1}$$

where α, α_1, β, and β_1 are arbitrary constants, which are determined by the specific details of the physical issue. The function $u(x, t)$ can be regarded as the displacement of the isopycnals, x is the horizontal coordinate, and t is time. Equation (1) includes the traditional quadratic (uu_x) and cubic nonlinearities (u^2u_x), as well as the third-order (u_{xxx}) and fifth-order linear dispersion (u_{xxxxx}) terms, similar to those found in previous research [12].

In the case of the Gardner-Kawahara equation, where some coefficients are set to be zero, (1) can be reduced to well-known equations such as, for instance, the Gardner equation as given in (2) and the Kawahara equation as in (3) [18].The Gardner equation, often referred to as the modified Korteweg-de Vries equation [7, 33], includes quadratic nonlinearity, cubic nonlinearity, and third-order linear dispersion, as discussed in previous studies [2, 15]. This equation has been studied in greater depth using the same model by [22, 24]. The wave model will generate solitons with various shapes as shown in the paper [6]. This paper will focus on both the Gardner equation and the Kawahara equation to observe the differences in solitary wave propagation between these two equations. Gardner equation can be written as,

$$u_t + \alpha u u_x + \alpha_1 u^2 u_x + \beta u_{xxx} = 0. \tag{2}$$

The quadratic and cubic nonlinear terms for the conditions of the stratified ocean contribute to the internal wave dynamics, depending on specific density and current stratification, where the initial-value problem for intense disturbances may be quite complicated [15]. We believe there are still many things to study about the influence of cubic nonlinear terms on ocean wave problems. In a different case, when the cubic nonlinear term is ignored by setting $\alpha_1 = 0$, (1) reduces to (3), known as the Kawahara equation, introduced by [17]. The Kawahara equation is a type of KdV equation that includes quadratic nonlinearity and third and fifth-order linear dispersion, as discussed in [5, 28]. Recent research on this model, such as the study by [16], highlights its growing interest.

Numerical simulations show that highly oscillatory behavior can be captured accurately which had been carried out by [17, 20] in solving (3),

$$u_t + \alpha u u_x + \beta u_{xxx} + \beta_1 u_{xxxxx} = 0. \tag{3}$$

Previous studies have shown that Kawahara equation is highly reliable to model many complex systems such as seas, oceans, and plasma physics [5, 13]. In this paper, we employ the Pseudospectral method to obtain the different wave propagations generated by both the Gardner and Kawahara equations. Notably, while previous studies utilized alternative methods, our approach extends the simulation up to $t = 100$, ensuring robustness and smooth results. In Section 2, we explain the Pseudospectral method has produce better results, while in Section 3, conclusions will be addressed.

2 Pseudospectral Method

In this section, we directly begin with (1), and subsequently set certain constants to zero in order to obtain the numerical results for the Gardner and Kawahara equations. Pseudospectral method is an alternative to the finite difference or finite element method to solve nonlinear partial differential equations claim by [9]. Previous studies have used this method to solve the Ostrovsky wave model problems, as seen in the works of [3, 4], which produced solitons with shapes that are specific to the Ostrovsky model. Generally, this method is simple to be implement and has much lower computing cost with the Fast Fourier Transform (FFT). FFT is known to be a very efficient algorithm for calculating the Discrete Fourier Transform (DFT). Essentially, DFT converts real space to Fourier space and substitutes the temporal derivative by finite difference approximation.

The infinite interval is replaced by $-L < x < L$ by applying a coordinate transformation which is a linear transformation to transform the spatial variable $u(x,t)$ from (1) to 2π periodicity-dependent variables $V(\xi,t)$. By introducing the linear transformation $\xi = sx + \pi$ where $s = \dfrac{\pi}{L}$, (1) can be written as,

$$V_t + \alpha s V V_\xi + \alpha_1 s V^2 V_\xi + \beta s^3 V_{\xi\xi\xi} + \beta_1 s^5 V_{\xi\xi\xi\xi\xi} = 0. \tag{4}$$

In the Pseudospectral method, the nonlinear terms, VV_ξ and V^2V_ξ, are computed in the real domain before transforming back to the Fourier space at each time step. By letting $W_1 = \dfrac{1}{2}V^2$ and $W_2 = \dfrac{1}{3}V^3$ for quadratic and cubic nonlinear terms, (4) becomes,

$$V_t + \alpha s W_{1\xi} + \alpha_1 s W_{2\xi} + \beta s^3 V_{\xi\xi\xi} + \beta_1 s^5 V_{\xi\xi\xi\xi\xi} = 0.$$

To obtain the numerical solution, the interval $[0, 2\pi]$ is discretized by $N+1$ equidistant points. Let $\xi_0 = 0$, ξ_1, ξ_2, $\ldots$, $\xi_N = 2\pi$, so that $\Delta\xi = \dfrac{2\pi}{N}$. In this case, N will always be even and is to be a power of two. Let say $a = \dfrac{N}{2}$. The DFT of $V(\xi_j,t)$, $W_1(\xi_j,t)$, and $W_2(\xi_j,t)$ for $j = 0,1,2,\ldots,N-1$ denoted by $\hat{V}(p,t)$, $\hat{W}_1(p,t)$, and $\hat{W}_2(p,t)$, respectively, are given by,

$$\hat{V}(p,t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v(\xi_j,t) e^{-\frac{2\pi j p}{N} i},$$

$$\hat{W}_k(p,t) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} w(\xi_j,t) e^{-\frac{2\pi j p}{N} i}, \qquad k = 1,2.$$

Therefore, the DFT equation with respect to ξ is given as follows,

$$\hat{V}_t(p,t) + i\alpha p \hat{W}_1(p,t) + i\alpha_1 p \hat{W}_2(p,t) - \beta p^3 s^3 \hat{V}(p,t) + \beta_1 i p^5 s^5 \hat{V}(p,t) = 0. \tag{5}$$

Then, by using the following central difference approximations for $\hat{V}_t$ and central averaging formula for $\hat{V}$, respectively, we have,

$$\hat{V}_t(p,t) \approx \frac{\hat{V}(p,t+\Delta t) - \hat{V}(p,t-\Delta t)}{2\Delta t},$$

$$\hat{V}(p,t) \approx \frac{\hat{V}(p,t+\Delta t) + \hat{V}(p,t-\Delta t)}{2}.$$

To simplify (5), by denoting $\hat{V}(p, t+\Delta t)$ and $\hat{V}(p, t-\Delta t)$ by $\hat{V}_{pt}$ and $\hat{V}_{mt}$, respectively, the scheme is given by,

$$\hat{V}_{pt} = \frac{1}{1 - ip^3 s^3 \Delta t\beta + i\beta_1 p^5 s^5}\left[(1 + is^3 p^3 \Delta t\beta - i\beta_1 p^5 s^5)\hat{V}_{mt} - 2ip\Delta t\left(\alpha \hat{W}_{1\xi}(p,t) + \alpha_1 \hat{W}_{2\xi}(p,t)\right)\right].$$

3 Results

Numerical results for Gardner equation (2) and Kawahara equation (3) are presented in this section. In this study, all computational work was done using a laptop computer with the following specification: Intel® i5-1135G7, 2.40GHz 4 cores and DDR-4 24GB memory. The C++ language was used to implementour algorithm while the visualization was done using an open-source software Gnuplot. We investigated the effects of cubic nonlinear and fifth-order linear dispersion terms in our study. Therefore, we could analyze the propagation of waves and their applications to surface and internal water waves.

Example 3.1. *In this computations, we set the values of computations as $N = 2048$, $L = 50$, and $\Delta t = 0.01$ to perform the numerical computations for Gardner equation (2). The initial condition for this computations is given by Kurkina et al. [20],*

$$U(x,0) = \frac{\sqrt{6V}}{2}\left[\tanh\left(\sqrt{\frac{V}{2}}x + \phi\right) - \tanh\left(\sqrt{\frac{V}{2}}x - \phi\right)\right],$$

where

$$\phi(V) = \frac{1}{4}\ln\left(\frac{1+\sqrt{6V}}{1-\sqrt{6V}}\right).$$

The result for Example 3.1 is as shown in Figure 1, and we can observe the formation of a KdV-like soliton formed at the initial time as a regular soliton. The value of V represents the size of the amplitude in the initial condition. When the value of V is increased from $V = 0.1$ to $V = 0.1663$ and $V = 0.166666$, we observed that both amplitude and width increase. Consequently, the tops become flatter, similar to the fat graphs studied by [32] and the fat solitons studied by [21]. Until at one point of $V = 0.166666$, the soliton becomes a table-top wave in the initial. When we increase the time, the soliton propagates in the positive (right) direction as shown in Figure 1. We can see that new solitons start to build at time $t = 10$ for $V = 0.1663$ and $V = 0.166666$. After that, at $t = 20$ for $V = 0.1663$, bi-solitons emerge, and for $V = 0.166666$, the solitons abruptly change their polarity to build triple-solitons.

Then, the number of solitons increases parallel with increasing time, called multi-solitons when $t = 20$, $t = 30$ and $t = 50$. The second solitons for $V = 0.166666$ are getting more and more similar to the first solitons at $V = 0.1663$, this shows that the solitons with $V = 0.166666$ move one solitons ahead compared to $V = 0.1663$ and move in the same direction and exist likely to merge. We do not rule out the possibility of perfect merging, double merging, imperfect merging and absorb-emit merging phenomena considering the nonlinear nature found in Gardner equation (2), as the studied in [35]. This pattern is not applied for $V = 0.1$, we can see that the regular solitons remain qualitatively the same with a stable and smooth solitary wave. When solitons remain qualitatively the same with low speed, moving to the right positive sign as time increases, we can see this shape of the wave in the previous study [25]. We can see that the soliton with the highest and widest amplitude ($V = 0.166666$) is faster than the soliton with a smaller width ($V = 0.1663$) and the soliton with the lowest amplitude and smallest size ($V = 0.1$), which is the slowest propagation. For Example 3.1, we observe that our results are in good agreement with previous studies done by Khusnutdinova et al. [18] and Kurkina et al. [20]. This result also shows the behavior of nonlinear waves as stated by Crighton [10] and Curry et al. [11] that nonlinear waves will break in multi-valuedness.

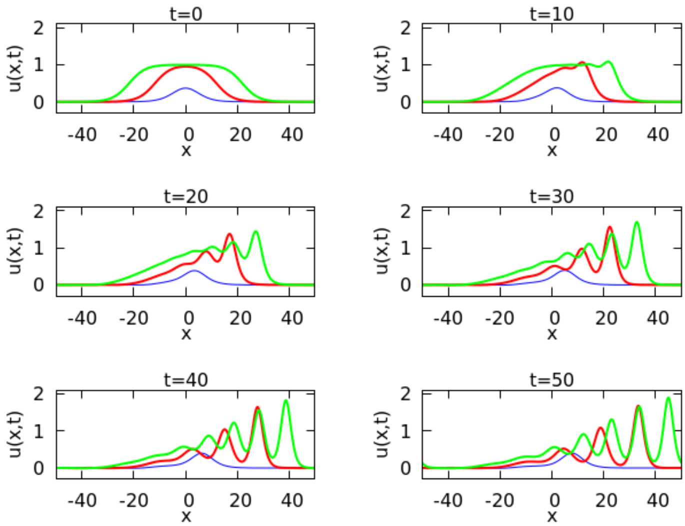


Figure 1: Snapshot for Example 3.1 , solution of Gardner equation (2) at $t = 0$, $t = 10$, $t = 20$, $t = 30$, $t = 40$ and $t = 50$, with $V = 0.1$ (blue line), $V = 0.1663$ (red line), and $V = 0.166666$ (green line).

Figure 2 shows the movement of wave at all time until $t = 100$ when $V = 0.1$. It was seen that when $t = 100$ is in the state of $V = 0.1$, the soliton moves by 10 units, which proves the role of V is related to the speed of the soliton which is $0.1\,\mathrm{ms}^{-1}$. This movement of the left wing produces a uniform wave without splitting making it strongly stable solitons with a constant velocity. This solitons behavior can be seen as quasi-stationary long-living solitary waves that permanently radiating from one side small-amplitude linear waves produce radiating solitons [8, 36].

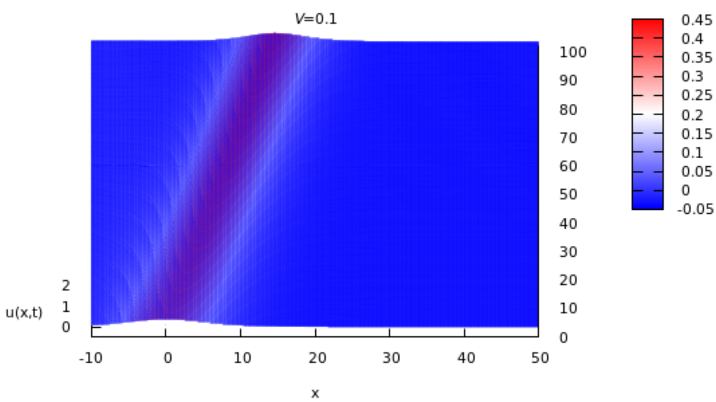


Figure 2: Example 3.1 , solution of Gardner equation (2) for all time t until $t = 100$, with $V = 0.1$.

Figure 3 shows the propagation of wave at all time until $t = 100$ when $V = 0.1663$. The soliton splits and becomes narrow, and its amplitude increases parallel to time until $t = 100$. It shows a different behavior than $V = 0.1$ when the soliton will break into three soliton, shows the first soliton is faster with a higher amplitude when it has exceeded 50 units at $t = 100$. The second soliton with smaller amplitude is slower by only 30 units at $t = 100$. Meanwhile, the third soliton that appears stationary resembles the soliton in Figure 2.

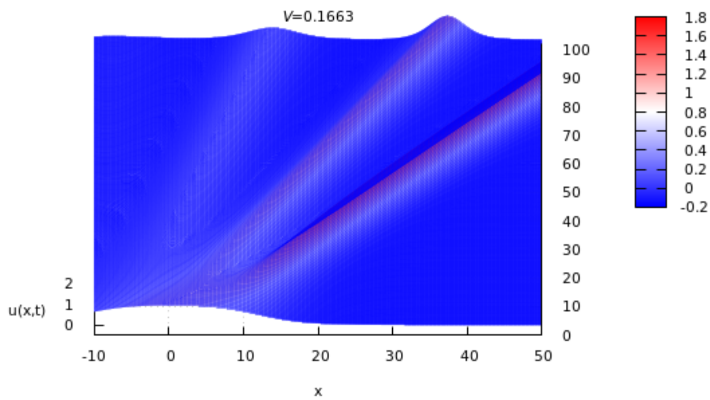


Figure 3: Example 3.1 , solution of Gardner equation (2) for all time t until $t = 100$, with $V = 0.1663$.

Figure 4 shows the movement of wave at all time until $t = 100$ when $V = 0.166666$. Observing the movement of soliton with $V = 0.166666$, the initial soliton is wider until it forms a table-top wave and it produces more solitons, known as multi-solitons. Figure 3 shows the solitons with the highest amplitude will move faster like the first soliton in Figure 4 when it exceeds 50 units with a time of only $t = 70$ with a speed limit of $0.7143\,\mathrm{ms}^{-1}$. It can also be seen that the second soliton is the same as the first soliton in Figure 3 and the last soliton which is the fifth soliton is similar to the stationary soliton in Figure 2. We think it is possible that the soliton that exists in the middle of the fifth and second soliton (fourth and third soliton) is a soliton that exists from the violation or combination of solitons. It explains how the width of the soliton affects the motion of the soliton. This is a new discovery about the behavior of fat solitons. It can be seen that the red color is fading as the solitons split into multi-solitons. A soliton that is split will lose energy as discussed by [14] proves that it is weak in the regime of high-quality splitting. Therefore, it is concluded that the fat soliton splits become narrower, and their amplitudes increase.

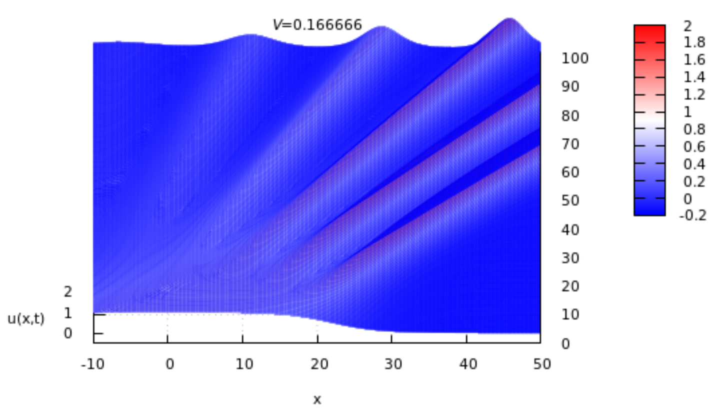


Figure 4: Example 3.1 , solution of Gardner equation (2) for all time t until $t = 100$, with $V = 0.166666$.

Example 3.2. *For Example 3.2, we have produced numerical results for Kawahara equation (2) with the fifth-order linear dispersion. We set the coefficient value as $\beta_1 = -1$. We set the values of $N = 2048$, $L = 100$ and $\Delta t = 0.01$. The initial condition for this case follows the one implemented by Khusnutdinova et al. [18],*

$$U(x,t) = -\frac{105}{169} B\, sech^4\left(\frac{x - Vt}{\Delta}\right),$$

where

$$B = -\frac{36}{169}V, \quad \Delta = \sqrt{-52B}, \quad and \quad B < 0.$$

Figure 5 shows that the KdV-like solitons maintain a constant width when the value of V increases, but the amplitude of the solitons simultaneously increases at $t = 0$. It is apparent that the amplitude of the solitons when $V = 0.5$ is lower than when $V = 0.6$, and when $V = 0.8$, solitons become much higher. Then, we analyse the wave propagation when time increases to $t = 10, 20, 30, 40$ and 50, respectively. We observe that at $t = 10$, the soliton with $V = 0.8$ precedes the motion of solitons. They travel to the right (positive) with difference speed base on difference value of V. The value of V that is high will move faster as analyzed from the movement when $t = 20$ and more longer than that. This movement are not interrupting the number of solitons or the regular shape of solitons. But one may observe the emergence of oscillatory tails. Solitons with oscillatory tails affecting how they interact with other solitons and propagate. The tails of these solitons oscillate without emitting radiation, creating a barrier between neighboring solitons, preventing their interaction [1].

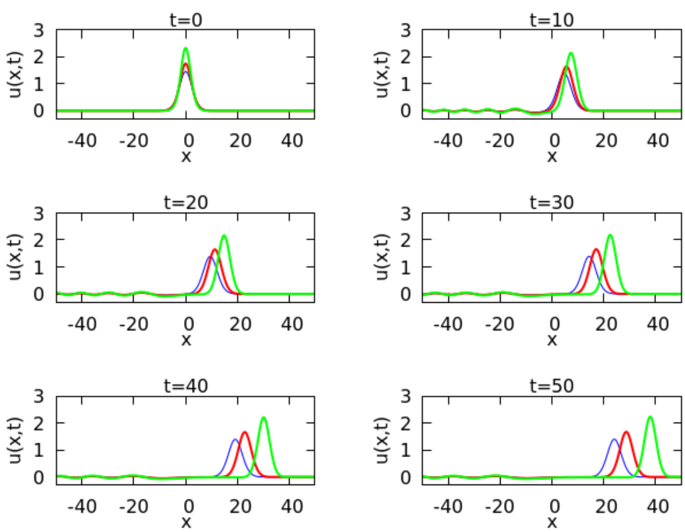


Figure 5: Snapshot for Example 3.2, solution of Kawahara equation (3) at $t = 0, t = 10, t = 20, t = 30, t = 40$ and $t = 50$, with $V = 0.5$ (blue line), $V = 0.6$ (red line), and $V = 0.8$ (green line).

We can see at $t = 50$, the waves move in unison to the right positive sign with respective decreasing speeds due to the weakening of the energy; it shows the dissipative wave behavior [11]. Obviously, we can see that the higher waves move faster than the lower ones and the moving soliton brings the oscillatory tails with stable regular soliton through time increases. Our result agrees with that of Gong et al. [13] and Aljahdaly et al. [5]. In addition, the Kawahara equation (3) produces solitary waves with oscillatory tails behaviors [18, 20].

When the time is extended to $t = 100$ as illustrated in Figure 6, a soliton with $V = 0.5$ moves 50 units without changing its size and remains a single solution with the oscillatory tails consistently until reaching $t = 100$ as shown in Figure 9(a). We can observe that the speed of the soliton is $0.5\,\text{ms}^{-1}$, which is half of the time. It proves the role of V in the soliton is the speed of the soliton. The soliton is a regular soliton that maintain its shape and speed during propagation and interactions with other solitons with strongly stability, preserving its shape and speed even after collisions with other solitons. This soliton is typically obtained using the inverse scattering transform, which is based on the integrability of the field equations that propagate at a constant velocity, determined by their initial shape and the specific model they are derived from. This type of soliton is also like quasi-stationary long-living solitary waves which eventually produce radiating solitons when permanently radiating from one side small-amplitude linear waves [8, 36]. The oscillatory tails as seen in Figure 5 are very small, we cannot see them in Figure 6 which considers a longer time.

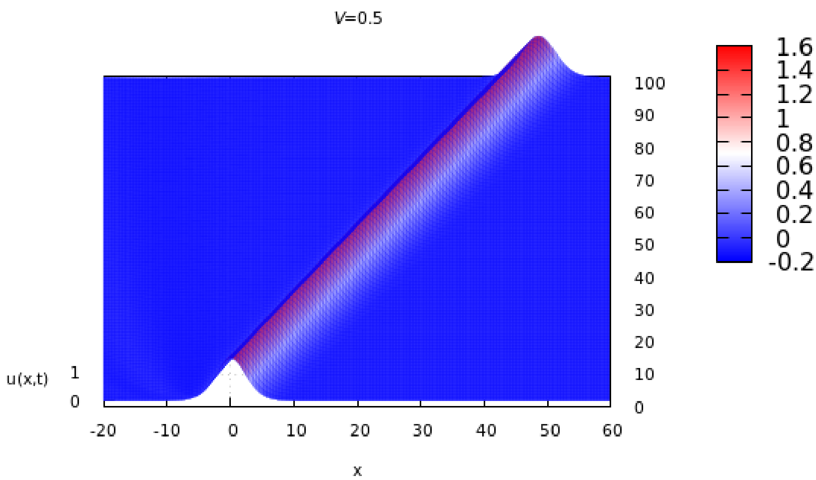


Figure 6: Example 3.2, solution of Kawahara equation (3) for all time t until $t = 100$, with $V = 0.5$.

Figure 7 shows that the soliton with $V = 0.6$ moves fast and reaches 50 units without changing size of soliton and still becomes a single solution with the oscillatory tails consistently at $t = 100$ as shown in Figure 9(b). This difference compared to Figure 6 explains how the role of velocity, V in the movement of the soliton with time. It can also be seen that the soliton amplitude is also higher compared to Figure 6. It is caused by a different initial shape as seen in Figure 5 when $t = 0$. From the point of view of shape, it is not much different from Figure 6 by displaying regular solitons that maintain their shape with strongly stability which is a natural property of linear waves [8, 36]. This is the same with a findings claim linear dispersive equations either disperse completely or in the presence of an external potential, decompose into a superposition of a radiative state that disperses to zero, plus a bound state that exhibits phase oscillation but is otherwise stationary [30].

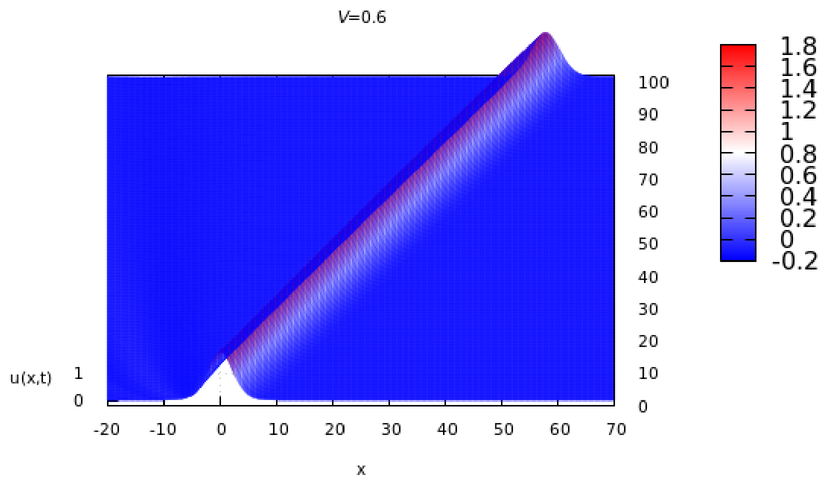


Figure 7: Example 3.2, solution of Kawahara equation (3) for all time t until $t = 100$, with $V = 0.6$.

Figure 8 shows that the soliton with $V = 0.8$ moves faster than $V = 0.5$ and $V = 0.6$. The speed trend of this soliton is parallel to the increasing value of V, when this soliton with $V = 0.8$ accelerates through 50 units in less than $t = 80$ without changing size of soliton and still becomes a single solution with the oscillatory tails consistently at $t = 100$ as shown in Figure 9(c). The amplitude of this soliton is also high as can be seen in Figure 8 up to 2.5. Just like Figures 6 and 7, the strong wave is maintained until it resembles a gap soliton. Gap solitons are characterized by their existence in the linear spectral band gap of the medium, and they arise due to the balance between dispersion and nonlinearity [23, 26]. Notably, despite these variations in speed, the solitons remain qualitatively the same.

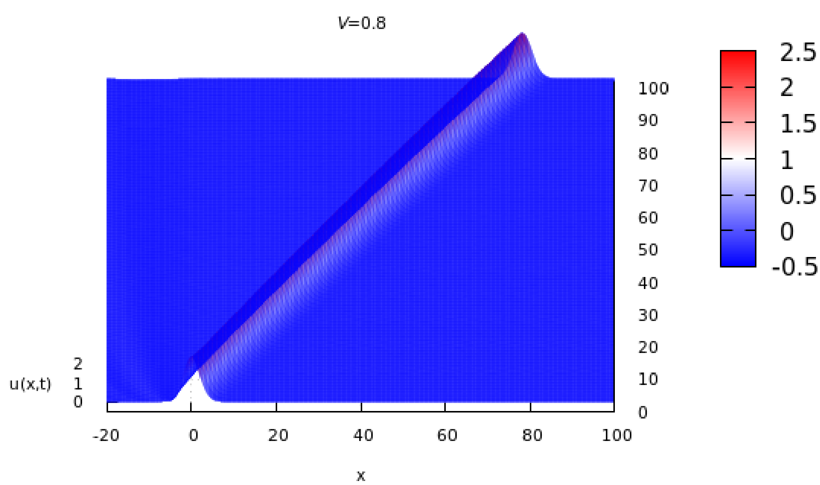
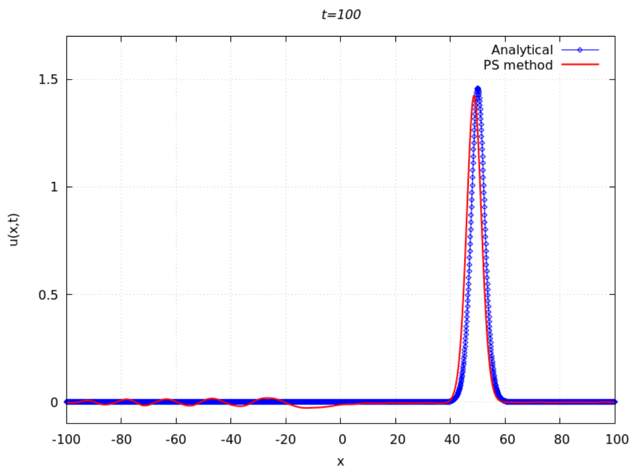
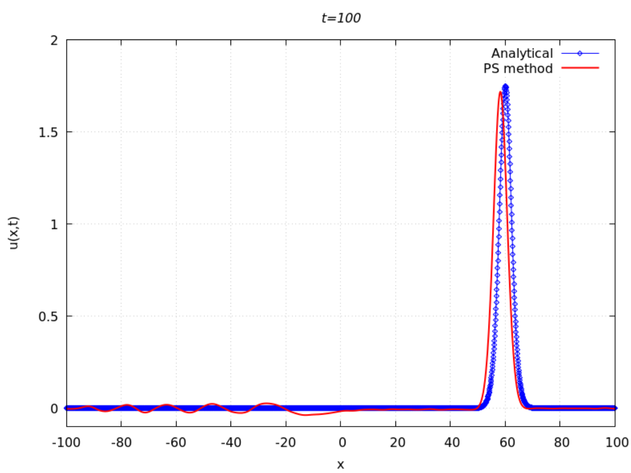


Figure 8: Example 3.2, solution of Kawahara equation (3) for all time t until $t = 100$, with $V = 0.8$.

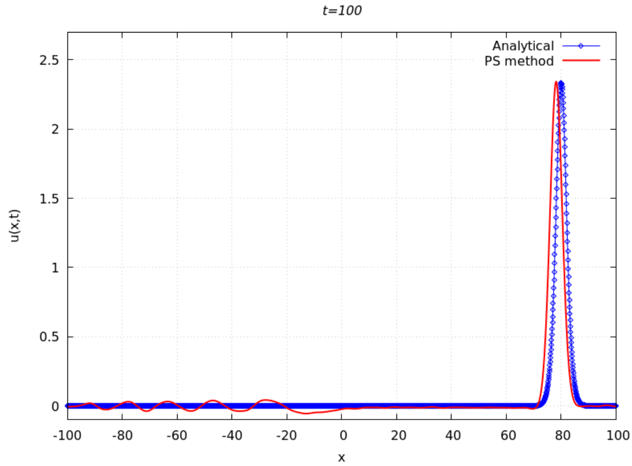
Comparing to Example 3.1, the Gardner equation solution does not have an exact solution, therefore we just do the comparison for the Kawahara equation problem, Example 3.2. We compare the analytical results with the results obtained using the PS method in Figure 9 at $t = 100$ for Kawahara equation (3) with different value of V. Solitary waves with oscillatory tail behaviors are produced in the numerical results but not in the analytical results. This outcome has been widely discussed by researchers, with analytical studies suggesting that exact solutions produce oscillatory tail behaviors, as demonstrated in the works of [19, 34] and more recently by Khusnutdinova et al. [18]. These efforts address higher-order nonlinear and dispersive terms, similar to the numerical tests conducted by Kawahara [17]. Therefore, our results are consistent with previous studies that claim the Kawahara equation exhibits oscillatory tails.



(a) Solution of Kawahara equation (3) at $t = 100$, with $V = 0.5$.



(b) Solution of Kawahara equation (3) at $t = 100$, with $V = 0.6$.



(c) Solution of Kawahara equation (3) at $t = 100$, with $V = 0.8$.

Figure 9: Comparison of exact and PS numerical solutions of the Kawahara equation (3) at $t = 100$ for different value of V.

4 Conclusion

In conclusion, the study successfully achieved its objectives by utilizing the PS method to do the simulation and observe the propagation of wave in extended time for both Gardner and Kawahara equations. The results illustrate distinct behaviors of solitary waves influenced by cubic nonlinearity and fifth-order linear dispersion terms. The Gardner equation demonstrates how solitary waves vary in size and can lead to multiple solitary waves over time, with changes in the parameter V that affecting their behavior. In contrast, the Kawahara equation consistently forms solitary waves with increasing amplitudes over time, influenced by value of V and we observed the existing of oscillating tails in the propagation. Both equations highlight the effects of higher-order terms, revealing complex dynamics between nonlinear and dispersive effects. The research enhances understanding of solitary wave propagation and internal wave interactions, aligning with the investigation into the impacts of cubic nonlinear and fifth-order linear dispersion terms.

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Conflicts of Interest The authors declare no conflict of interest.

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